A_∞-OPERADS AND MAY'S RECOGNITION PRINCIPLE

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These are the notes for a talk I gave at the Graduate Student Seminar in April, 2025 at the University of Massachusetts Amherst.

1. Introduction

Often in mathematics we encounter structures that admits a multiplication i.e. a binary operation. Given a space with a multiplication, the basic questions we can ask for are whether it is unital, is it associative, is it commutative? The class of spaces that have all of these desired properties are commutative topological monoids. In the top of that, if the points in the space admits inverses as well, then we have the notion of commutative topological groups. But there are spaces which are very close to being a topological group, but not exactly. One such space is the loop space ΩX associated to a topological space X which is the central object of this talk. In algebraic topology one of the main ways to probe a space is by looking at its loop space. Loop spaces are important objects in algebraic topology for a variety of reasons, including their connections to higher homotopy groups, spectra, and generalized cohomology theories.

2. Loop spaces

Let $X = (X, *)$ be a compactly generated weakly Hausdorff based topological space, and $\Omega X := \mathbf{Top}_*((\mathbb{S}^1, 1), (X, *))$ be the (1-fold) loop space of X at $*$ with the compact open topology. From a first course in algebraic topology, one learns that there is a multiplication operation on ΩX viz.

$$\mu : \Omega X \times \Omega X \to \Omega X$$

defined as,

$$\mu(\alpha, \beta) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

concatenating the loops α followed by β [1]. There is this standard observation that

Observation 2.0.1. The multiplication μ is not associative on the nose, on ΩX.

Reason 2.0.2. $(\alpha \cdot \beta) \cdot \gamma \neq \alpha \cdot (\beta \cdot \gamma)$ even as set maps. In other words, $\mu(\mu(-,-),-)$ and $\mu(-,\mu(-,-))$ are different ternary operations on ΩX.

Conclusion 2.0.3. In fancy terms, with the chosen binary operation on ΩX, the space of ternary operations on ΩX fails to be a singleton space.

Having that, next one must be curious about is whether $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

Proposition 2.0.4. *For $\alpha, \beta, \gamma \in \Omega X$, $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$ i.e. there exists a map $\mathcal{K}(3)$[2] $\times (\Omega X)^3 \to \Omega X$ satisfying the boundary conditions, where $\mathcal{K}(3)$ is a space that looks like*

Date: March 2025.

[1] Henceforth, we shall write $\alpha \cdot \beta$ for $\mu(\alpha, \beta)$.

Figure 1. $\mathcal{K}(3)$

Proof.

$$H(t,s) = \begin{cases} \alpha((2s)t + (4s)(1-t)), & 0 \le s \le (\frac{t}{2} + \frac{1-t}{4}) \\ \beta(4s-2), & (\frac{t}{2} + \frac{1-t}{4}) \le s \le (\frac{3t}{4} + \frac{1-t}{2}) \\ \gamma((4s-3)t + (2s-1)(1-t)), & (\frac{3t}{4} + \frac{1-t}{2}) \le s \le 1 \end{cases}$$

□

We say μ is homotopy associative i.e. associative upto coherent homotopy.

Proposition 2.0.5. *μ is homotopy unital i.e. there exists homotopies $\mu(* \times 1) \simeq 1 \simeq \mu(1 \times *)$.*

Corollary 2.0.6. *ΩX is a homotopy associative H-space.*

One thing we notice here is that with a choice of multipliation μ on ΩX i.e. a map $\mathcal{K}(2) \times (\Omega X)^2 \to \Omega X$, where $\mathcal{K}(2)$ is a space that looks like

Figure 2. $\mathcal{K}(2)$

the two point space $\{\mu(1 \times \mu), \mu(\mu \times 1)\}$ is contractible inside the space of ternary operations on ΩX i.e. there is a path between $\mu(1 \times \mu)$ and $\mu(\mu \times 1)$ in $\mathcal{E}nd_X(3)$[3]. Moving forward, there are essentially five ways to multiply four loops $\alpha,\ \beta,\ \gamma,\ \delta$ in order, viz.

$$((\alpha\beta)\gamma)\delta \quad (\alpha(\beta\gamma))\delta \quad (\alpha\beta)(\gamma\delta) \quad \alpha(\beta(\gamma\delta)) \quad \alpha((\beta\gamma)\delta)$$

with not only just homotopies between any pair from the list, but also higher homotopies.

Proposition 2.0.7. *Any pair from the above list are not equal on the nose, rather are homotopic.*

Proposition 2.0.8. *The following pentagon*

$$\begin{tikzcd}
 & ((\alpha\beta)\gamma)\delta \arrow[dl] \arrow[dr] & \\
(\alpha\beta)(\gamma\delta) \arrow[d] & & (\alpha(\beta\gamma))\delta \arrow[d] \\
\alpha(\beta(\gamma\delta)) & & \alpha((\beta\gamma)\delta) \arrow[ll]
\end{tikzcd}$$

is not commutative.

Proposition 2.0.9. *There exists a 2-homotopy i.e. a map $\mathcal{K}(4) \times (\Omega X)^{\times 4} \to \Omega X$ which fills in the above pentagon, where $\mathcal{K}(4)$ is a space which looks like*

Moving on, it is very tempting to claim that $\mathcal{K}(5)$ is a space that looks like a regular polyhedron with 14-vertices, with pentagonal faces, i.e. 8 faces and 20 edges

Exercise 2.0.10. Though it satisfies Euler's formula $V - E + F = 2$, such regular polyhedron doesn't exist.

[3]We will introduce the notation in the next section.

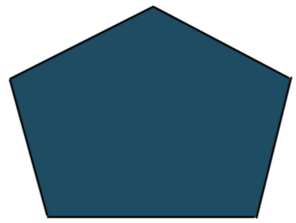


FIGURE 3. $\mathcal{K}(4)$

Reason 2.0.11. Observe that we have the following non-commutative square

$$\begin{tikzcd}
((\alpha_1 \cdot \alpha_2) \cdot \alpha_3)(\alpha_4 \cdot \alpha_5) \arrow[r] & (\alpha_1 \cdot (\alpha_2 \cdot \alpha_3))(\alpha_4 \cdot \alpha_5) \arrow[d] \\
(((\alpha_1 \cdot \alpha_2) \cdot \alpha_3) \cdot \alpha_4) \cdot \alpha_5 \arrow[r] \arrow[u] & ((\alpha_1 \cdot (\alpha_2 \cdot \alpha_3)) \cdot \alpha_4) \cdot \alpha_5
\end{tikzcd}$$

that is filled by a 2-homotopy which looks like

FIGURE 4. This is not $\mathcal{K}(4)$

and it is clearly not $\mathcal{K}(4)$. Rather $\mathcal{K}(5)$ is a space that looks like

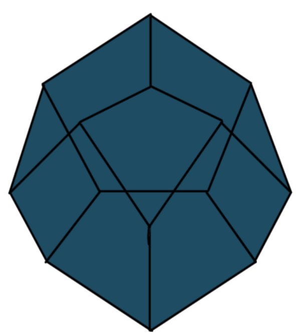


FIGURE 5. $\mathcal{K}(5)$

This shows the complex combinatorics of $\mathcal{K}(n)$ is general.

But what do we take away from these low dimensional cases? We observe in this section that these spaces $\mathcal{K}(n)$ are sort of the abstraction of a family of composable loops in X, useful for the "bookkeeping" and applications of such families. This motivates us to learn what an *operad* is!

3. Non-symmetric operad

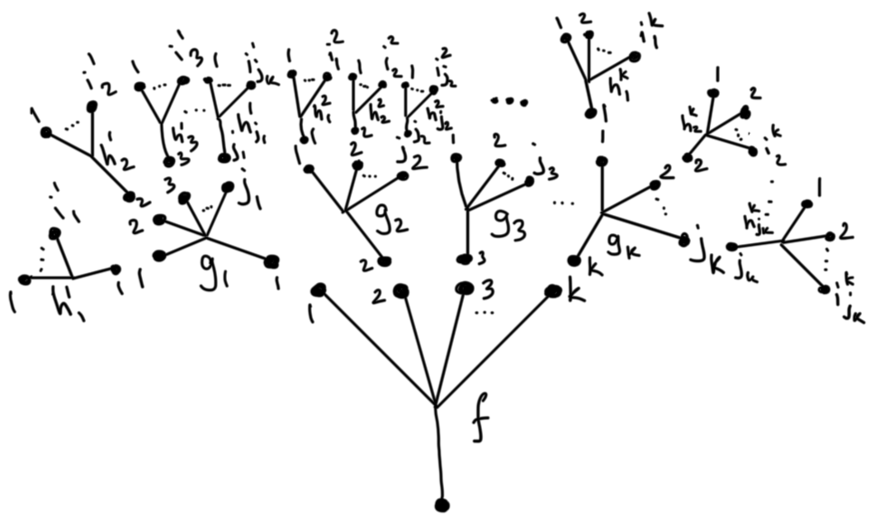
A non-symmetric topological operad $\mathcal{O}$ is a collection $\{\mathcal{O}(n)\}_{n=0}^{\infty}$ of spaces, with $\mathcal{O}(0)$ a singleton space, together with the following data:

(i) Continuous maps (abusing notation),

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \mathcal{O}(j_2) \times \cdots \times \mathcal{O}(j_k) \to \mathcal{O}(j_1 + j_2 + \cdots + j_k)$$

such that if there are continuous maps

$$\gamma : \mathcal{O}(j_1) \times \mathcal{O}(i_1^1) \times \mathcal{O}(i_2^1) \times \cdots \times \mathcal{O}(i_{j_1}^1) \to \mathcal{O}(i_1^1 + \cdots + i_{j_1}^1)$$
$$\gamma : \mathcal{O}(j_2) \times \mathcal{O}(i_1^2) \times \mathcal{O}(i_2^2) \times \cdots \times \mathcal{O}(i_{j_2}^2) \to \mathcal{O}(i_1^2 + \cdots + i_{j_2}^2)$$
$$\vdots$$
$$\gamma : \mathcal{O}(j_k) \times \mathcal{O}(i_1^k) \times \mathcal{O}(i_2^k) \times \cdots \times \mathcal{O}(i_{j_k}^k) \to \mathcal{O}(i_1^k + \cdots + i_{j_k}^k)$$



we have

$$\gamma(\gamma(f; g_1, \cdots, g_k); h_1^1, \cdots, h_{j_k}^k) = \gamma(f; e_1, \cdots, e_k)$$

where

$$e_s = \gamma(g_s; h_1^s, \cdots, h_{j_s}^s)$$

(ii) An identity element $1 \in \mathcal{O}(1)$ such that

$$\gamma(1; d) = d$$

for all $d \in \mathcal{O}(j)$ and

$$\gamma(c; \underbrace{1, 1, \cdots, 1}_{k \text{ many}}) = c$$

for all $c \in \mathcal{O}(k)$.

Example 3.0.1. Let X be a compactly generated Hausdorff topological space. We define the non-symmetric *endomorphism operad* $\mathcal{E}nd_X$ as follows. Let $\mathcal{E}nd_X(n)$ denote the space of based continuous maps $\mu : X^{\times n} \to X$, with $X^{\times 0} = *$, and $\mathcal{E}nd_X(0)$ is the singleton space $\{i : * \to X\}$. The data is defined by

(i) $\gamma(f; g_1, g_2, \cdots, g_k) = f(g_1 \times g_2 \times \cdots \times g_k)$ for $f \in \mathcal{E}nd_X(k)$ and $g_k \in \mathcal{E}nd_X(j_k)$.

(ii) The identity element $1 \in \mathcal{E}nd_X(1)$ is the identity map $\mathrm{id}_X : X \to X$.

Example 3.0.2. We define the non-symmetric *associative operad* $\mathcal{A}ss$ as follows. Let $\mathcal{A}ss(n)$ denote the singleton space $\{*\}$ for all $n \geq 0$. The data is defined by

(i) $\gamma(*; *, *, \cdots, *) = *$.

(ii) The identity element $1 \in \mathcal{A}ss(1)$ is $*$.

Just as group theory without representations is rather sterile, so operads are best appreciated by their representations, known as algebras, especially algebras with higher homotopies.

Definition 3.0.3. An action of an operad on a space X is a collection of maps $\psi_k : \mathcal{O}(k) \to \mathcal{E}nd_X(k)$ such that

(i) $\psi_1(1) = \mathrm{id}_X : X \to X$.

(ii) The following square

$$\begin{tikzcd}
\mathcal{O}(k) \times \mathcal{O}(j_1) \times \mathcal{O}(j_2) \times \cdots \times \mathcal{O}(j_k) \arrow[r, "\gamma"] \arrow[d, "\psi_k \times \psi_{j_i} \times \cdots \times \psi_{j_k}"'] & \mathcal{O}(j_1 + j_2 + \cdots + j_k) \arrow[d, "\psi_{j_1 + j_2 + \cdots + j_k}"] \\
\mathcal{E}nd_X(k) \times \mathcal{E}nd_X(j_1) \times \mathcal{E}nd_X(j_2) \times \cdots \times \mathcal{E}nd_X(j_k) \arrow[r, "\gamma"'] & \mathcal{E}nd_X(j_1 + j_2 + \cdots + j_k)
\end{tikzcd}$$

commutes.

Definition 3.0.4. A based space X is an $\mathcal{O}$-algebra if $\mathcal{O}$ acts on X.

Example 3.0.5. We shall characterize the non-symmetric $\mathcal{A}ss$-algebras. An action of $\mathcal{A}ss$ on a space X picks out a n-ary operation for each n. Suppose $\psi_2(*) = \mu : X^{\times 2} \to X$, then consider the commutative square

$$\begin{tikzcd}
\mathcal{A}ss(2) \times \mathcal{A}ss(1) \times \mathcal{A}ss(2) \arrow[r, "\gamma"] \arrow[d, "\psi_2 \times \psi_1 \times \psi_2"'] & \mathcal{A}ss(3) \arrow[d, "\psi_3"] \\
\mathcal{E}nd_X(2) \times \mathcal{E}nd_X(1) \times \mathcal{E}nd_X(2) \arrow[r, "\gamma'"'] & \mathcal{E}nd_X(3)
\end{tikzcd}$$

which forces $\psi_3(*) = \mu(1 \times \mu)$. Similarly, the commutativity of the square

$$\begin{tikzcd}
\mathcal{A}ss(2) \times \mathcal{A}ss(2) \times \mathcal{A}ss(1) \arrow[r, "\gamma"] \arrow[d, "\psi_2 \times \psi_2 \times \psi_1"'] & \mathcal{A}ss(3) \arrow[d, "\psi_3"] \\
\mathcal{E}nd_X(2) \times \mathcal{E}nd_X(2) \times \mathcal{E}nd_X(1) \arrow[r, "\gamma'"'] & \mathcal{E}nd_X(3)
\end{tikzcd}$$

forces $\psi_3(*) = \mu(\mu \times 1)$. Thus, μ is associative on the nose. Next, consider the commutative sqaure

$$\begin{tikzcd}
\mathcal{A}ss(2) \times \mathcal{A}ss(0) \times \mathcal{A}ss(1) \arrow[r, "\gamma"] \arrow[d, "\psi_2\times\psi_0\times\psi_1"'] & \mathcal{A}ss(1) \arrow[d, "\psi_1"] \\
\mathcal{E}nd_X(2) \times \mathcal{E}nd_X(0) \times \mathcal{E}nd_X(1) \arrow[r, "\gamma'"'] & \mathcal{E}nd_X(1)
\end{tikzcd}$$

which implies $1 = \mu(* \times 1)$. Similarly, the commutativity of the square

$$\begin{tikzcd}
\mathcal{A}ss(2) \times \mathcal{A}ss(1) \times \mathcal{A}ss(0) \arrow[r, "\gamma"] \arrow[d, "\psi_2\times\psi_1\times\psi_0"'] & \mathcal{A}ss(1) \arrow[d, "\psi_1"] \\
\mathcal{E}nd_X(2) \times \mathcal{E}nd_X(1) \times \mathcal{E}nd_X(0) \arrow[r, "\gamma'"'] & \mathcal{E}nd_X(1)
\end{tikzcd}$$

implies $1 = \mu(1 \times *)$. Thus, μ is unital. Therefore, X is an $\mathcal{A}ss$-algebra iff X has an unital and associative multiplication.

Observation 3.0.6. ΩX is not an $\mathcal{A}ss$-algebra.

Then the natural question here is that, can ΩX be realized as an algebra of some operad O? It seems that we have a guess of one such candidate.

Example 3.0.7. The spaces $\mathcal{K}(n)$ for $n \geq 0$ discussed in section 2, collectively forms an operad known as the *Stasheff associahedron operad* $\mathcal{K}$.

Fact 3.0.8. $\mathcal{K}(n)$ is a space which for all $n \geq 2$ looks like

(i) $\mathcal{K}(n)$ is homeomrphic to $\mathbb{I}^{n-2}$.

(ii) $\mathcal{K}(n)$ is a regular polyhedron with $\dfrac{n(n-1)}{2} - 1$ faces and $C_{n-1} = \dfrac{1}{n}\binom{2(n-1)}{n-1}$ vertices.

Proposition 3.0.9. *The loop space* ΩX *is a* $\mathcal{K}$*-space.*

4. A_∞-operad

One important thing to notice is that $\mathcal{K}(n)$ is contractible for all n, and thus our next definition.

Definition 4.0.1. A non-symmetric operad O is an A_∞-operad if $O(n)$ is contractible for all $n \geq 0$.

Definition 4.0.2. A A_∞-space is a space together with an action of an A_∞-operad.

Example 4.0.3. The Stasheff operad $\mathcal{K}$ is an A_∞-operad since $\mathcal{K}(n)$ is homeomorphic to $\mathbb{I}^{n-2}$ for all $n \geq 2$. Thus, a $\mathcal{K}$-space is an A_∞-space. It turns out that the converse is true as well.

Proposition 4.0.4. *An* A_∞*-space is a* $\mathcal{K}$*-space.*

Corollary 4.0.5. *The loop space* ΩX *is an* A_∞*-space.*

Another thing to notice about ΩX is that by definition $\pi_0(\Omega X) =: \pi_1 X$, which is a group. Thats motivates our next definition of a group-like space.

Definition 4.0.6. A space X is a *group-like* space if $\pi_0 X$ is a group.

Now the question arises what are group-like A_∞-spaces, other than the loop space ΩX, but the *Recognition principle*[4] due to May tells us that they are the only ones. In fancy terms, any group-like space is *weakly deloopable*.

Theorem 4.0.7. *Every group-like A_∞-space is weak homotopy equivalent to ΩX for some space X.*

5. Homotopy commutativity

We again recall the standard multiplication of two loops $\alpha, \beta \in \Omega X$,

$$\alpha \cdot \beta = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Observation 5.0.1. The multiplication is neither commutative nor homotopy commutative, on ΩX. This means that neither $\mu(-,-)$ and $\mu(\tau(-,-))$ are equal nor are they path-connected in $\mathcal{E}nd_X(2)$. This amounts to say that we have a map $\mathcal{K}^\Sigma(2) \to \mathcal{E}nd_X(2)$, where $\mathcal{K}^\Sigma(2)$ is a doubleton space. Apriori, it is not clear why this map picks up $\mu(-,-)$ and $\mu(\tau(-,-))$.

Definition 5.0.2. A symmetric topological operad O is a collection $\{O(n)\}_{n=0}^\infty$ of spaces, with $O(0)$ a singleton space, together with the following data:

(i) The data of a non-symmetric operad.

(ii) A right free action of S_k on $O(k)$ such that the following diagram

$$\begin{tikzcd} O(k) \times O(j_1) \times \cdots \times O(j_k) \arrow[rr, "1\times\rho"] \arrow[d, "\sigma\times 1"] & & O(n) \times O(j_{\sigma^{-1}(1)}) \times \cdots \times O(j_{\sigma^{-1}(k)}) \arrow[d, "\gamma"'] \\ O(k) \times O(j_1) \times \cdots \times O(j_k) \arrow[dr, "\gamma"'] & & O(j_1 + \cdots + j_k) \arrow[dl, "\sigma_{j_1,\cdots,j_k}"] \\ & O(j_1 + \cdots + j_k) & \end{tikzcd}$$

commutes.

Example 5.0.3. The symmetric endomorphism operad $\mathcal{E}nd_X^\Sigma$ where the data is same as the non-symmetric version along with the S_k action on $\mathcal{E}nd_X^\Sigma(k)$ given by $(f\sigma)(x_1, \cdots, x_k) = f(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(k)})$ for $f \in \mathcal{E}nd_X^\Sigma(k)$, $\sigma \in S_k$.

Definition 5.0.4. An action of an operad on a space X is a collection of S_k -equivariant maps $\psi_k : O(k) \to \mathcal{E}nd_X(k)$ such that

(i) $\psi_1(1) = \mathrm{id}_X : X \to X$.

(ii) The following square

$$\begin{tikzcd} O(k) \times O(j_1) \times O(j_2) \times \cdots \times O(j_k) \arrow[r, "\gamma"] \arrow[d, "\psi_k \times \psi_{j_i} \times \cdots \times \psi_{j_k}"'] & O(j_1 + j_2 + \cdots + j_k) \arrow[d, "\psi_{j_1+j_2+\cdots+j_k}"] \\ \mathcal{E}nd_X(k) \times \mathcal{E}nd_X(j_1) \times \mathcal{E}nd_X(j_2) \times \cdots \times \mathcal{E}nd_X(j_k) \arrow[r, "\gamma"'] & \mathcal{E}nd_X(j_1 + j_2 + \cdots + j_k) \end{tikzcd}$$

commutes.

[4]This is the base case of the Recognition principle

Example 5.0.5. We define the commutative operad $Comm$ similar to the associative operad $\mathcal{A}ss$ as follows. Let $Comm(n)$ denote the singleton space $\{*\}$ for all $n \geq 0$. The data of the operad is same as that of $\mathcal{A}ss$, together with the trivial action of S_k on $Comm(k)$. The equivariance and the structure maps characterizes the $Comm$-algebras, which are precisely spaces X having unital, associative and commutative multiplication.

6. Little n-cubes operad

We will deal with $n = 1$ case for this talk. The goal of this section is to understand how little cube operad E_1 acts on spaces.

Definition 6.0.1. The little cube operad E_1 is defined as follows. A point in $E_1(k)$ is a map

$$f : \underbrace{\mathbb{I} \amalg \mathbb{I} \amalg \cdots \amalg \mathbb{I}}_{k \text{ many}} \to \mathbb{I}$$

that specifies k disjoint little cubes in $\mathbb{I}$ i.e. f can be thought of as a k-tuple $(f_1, f_2, \cdots, f_k)$ of mutually disjoint little cubes inside $\mathbb{I}$, $f_k : \mathbb{I} \to \mathbb{I}$ being a nice map.

Proposition 6.0.2. *E_1 is a weak symmetric A_∞-operad i.e. an E_1-operad is weakly equivalent to an A_∞-operad.*

Corollary 6.0.3. *An E_1-space is weak homotopy equivalent to an A_∞-space.*

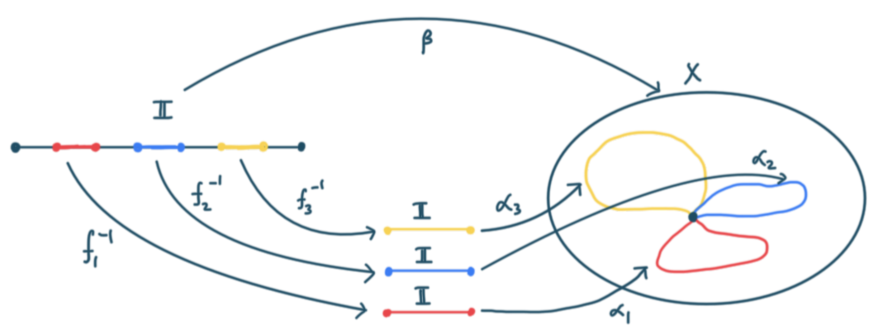
Next, we show that E_1 acts on ΩX.

Proposition 6.0.4. *ΩX is an E_1-space.*

Proof. The loop space ΩX can be viewed as the space of maps $(\mathbb{I}, \partial\mathbb{I}) \to (X, *)$ in the category of pairs of spaces. For a k-tuple $(f_1, f_2, \cdots, f_k)$ of mutually disjoint little cubes and k-many loops $(\alpha_1, \alpha_2, \cdots, \alpha_k) \in (\Omega X)^k$, $\alpha_k : (\mathbb{I}, \partial\mathbb{I}) \to (X, *)$. Then we obtain another loop

$$\beta : s \mapsto \begin{cases} \alpha_k(f_k^{-1}(s)) & s \in f_i(\mathbb{I}) \\ * & \text{otherwise} \end{cases}$$

This is the ordinary concatenation of loops in order, in disguise. The figure below makes it clear.



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Theorem 6.0.5. *Every group-like E_1-space is weak homotopy equivalent to ΩX for some space X.*

This can be generalized to n-fold loop spaces $\Omega^n X$.

Theorem 6.0.6. *Every group-like E_n-space is weak homotopy equivalent to $\Omega^n X$ for some space X.*

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